

OSCILLATING OPERATORS IN BILATERAL GRAND LEBESGUE SPACES

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Abstract.

In this paper we obtain the non - asymptotic estimations for oscillating integral operators in the so - called Bilateral Grand Lebesgue Spaces. We also give examples to show the sharpness of these inequalities.

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1. INTRODUCTION

The linear integral operator $T_\lambda f(x)$, or, more precisely, the *family of operators* of a view

$$T_\lambda f(x) = \int_{R^d} \exp(i\lambda\Phi_1(x, y)) \Phi_2(x, y) f(y) dy \quad (0)$$

will be called *oscillating*, if λ is real "great" number: $\lambda > 2$, $\lambda \rightarrow \infty$; $\Phi_2(x, y)$ is a fixed non - zero smooth function: $\Phi_2(\cdot, \cdot) \in C_\infty^0$ with finite support:

$$\exists C = \text{const} \in (0, \infty), \Phi_2(x, y) = 0 \quad \forall (x, y) : x^2 + y^2 \geq C, \quad (1)$$

$\Phi_1(x, y)$ is a fixed smooth function: $\Phi_1 \in C_\infty^0(R^d \times R^d)$ such that

$$\det \left(\frac{\partial^2 \Phi_1}{\partial x_i \partial y_j} \right) \neq 0 \quad (2)$$

on the support of the function $\Phi_2(\cdot, \cdot)$.

These operators are used in the theory of Fourier transform, theory of PDE, probability theory (study of characteristic functions and spectral densities) etc.

In the physical applications the function Φ_1 is called ordinary as *Phase function*, and the second function is named usually *Amplitude function*.

The behavior of the function $\lambda \rightarrow T_\lambda f$ as $\lambda \rightarrow \infty$ in the case when the function $f(\cdot)$ is smooth, is described in the so - called stationary phase method (on the other words, saddle - point method).

We denote as usually

$$|f|_p = \left(\int_{R^d} |f(x)|^p dx \right)^{1/p}; \quad f \in L_p \Leftrightarrow |f|_p < \infty.$$

We will consider further only the values p from the open interval $p \in (1, 2)$ and denote $q = q(p) = p/(p - 1)$; evidently, $q \in (2, \infty)$.

It is proved by E.M.Stein, see, e.g. in the book [17], p. 307 - 355 that the following estimation holds for the oscillating integral operator (0):

$$|T_\lambda f|_q \leq A(\Phi_1, \Phi_2) \lambda^{-d/q} |f|_p, \quad (3)$$

$A(\Phi_1, \Phi_2) \in (0, \infty)$.

Our aim is a generalization of estimation (3) on the so - called Bilateral Grand Lebesgue Spaces $BGL = BGL(\psi) = G(\psi)$, i.e. when $f(\cdot) \in G(\psi)$.

We recall briefly the definition and needed properties of these spaces. More details see in the works [3], [4], [5], [6], [12], [13], [9], [15], [16] etc. More about rearrangement invariant spaces see in the monographs [1], [10].

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a+0)$ and $\psi(b-0)$, with conditions $\inf_{p \in (a,b)} \psi(p) > 0$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b) = \Psi(a, b)$.

The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $h : R^d \rightarrow \mathbb{R}$ endowed with the norm

$$||h||G(\psi) \stackrel{def}{=} \sup_{p \in (a,b)} \left[\frac{|h|_p}{\psi(p)} \right]. \quad (4)$$

The $G(\psi)$ spaces with $\mu(X) = 1$ appeared in [9]. They are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [2], [15].

It was proved also that in this case each $G(\psi)$ space coincides with certain exponential Orlicz space, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

Remark 1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$||f||G(\psi_r) = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical Lebesgue spaces L_r .

We recall the expression for the fundamental function for $G(\psi)$ spaces. Namely,

$$\phi(G(\psi; a, b); \delta) = \sup_{p \in (a,b)} \left[\frac{\delta^{1/p}}{\psi(p)} \right]. \quad (5)$$

More information about the fundamental function for $G(\psi)$ spaces see in the article [13]; there was considered, in particular, many examples of $G(\psi)$ spaces with exact calculation of their fundamental functions.

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [3], [5], theory of probability in Banach

spaces [11], [9], [12], in the modern non-parametrical statistics, for example, in the so-called regression problem [12].

The article is organized as follows. In the next section we obtain the main result: upper bounds for oscillating operators in the Bilateral Grand Lebesgue spaces. In the last section we study the sharpness of the obtained results by the building of the suitable examples.

We use symbols $C(X, Y)$, $C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \rightarrow R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol \sim will denote usual equivalence in the limit sense.

2. MAIN RESULT: UPPER ESTIMATIONS

Let $\psi(\cdot) \in \Psi(a, b)$, where $1 \leq a < b \leq 2$. We define for the values $\lambda \geq 1$ and the values $q \in (b/(b-1), a/(a-1))$, where by definition at $a = 1 \Rightarrow a/(a-1) = +\infty$:

$$\psi^{(\lambda)}(q) = \lambda^{-d/q} \cdot \psi(q/(q-1)), \quad (6)$$

and define for the non - zero functions f belonging to the space $G(\psi)$

$$Z(\lambda, \psi, f) = \frac{\|T_\lambda f\| G(\psi^{(\lambda)})}{\|f\| G(\psi)}. \quad (7)$$

Theorem 1.

$$\sup_{\lambda \geq 1} \sup_{\psi \in \Psi(a, b)} \sup_{f \in G(\psi), f \neq 0} Z(\lambda, \psi, f) \leq A(\Phi_1, \Phi_2) < \infty. \quad (8).$$

Proof. Denote for the simplicity $u = T_\lambda f$; $u : R^d \rightarrow R$. We can assume without loss of generality that $\|f\| G(\psi) \leq 1$; this means that

$$\forall p \in (a, b) \Rightarrow |f|_p \leq \psi(p).$$

Using the inequality (3), we obtain the estimation

$$|u|_q \leq A(\Phi_1, \Phi_2) \lambda^{-d/q} \psi(p) = A(\Phi_1, \Phi_2) \lambda^{-d/q} \psi(q/(q-1)) = A(\Phi_1, \Phi_2) \psi^{(\lambda)}(q). \quad (9)$$

The assertion of theorem 1 follows after dividing on the $\psi^{(\lambda)}(q)$, tacking the maximum on the q and on the basis of the definition of the $G(\psi)$ spaces. \square

Now we offer the another version of upper estimations for oscillating operator in the Bilateral Grand Lebesgue spaces. Let $\psi(\cdot), \nu(\cdot), \zeta(\cdot)$ be three functions from the space $\Psi(a, b)$, $1 \leq a < b \leq 2$, such that

$$\nu(p) = \psi(p) \cdot \zeta(p). \quad (10)$$

Let us denote $\nu^*(q) = \nu(q/(q-1))$, $q \in (b/(b-1), a/(a-1))$.

Theorem 2.

$$\lambda^d \|T_\lambda f\| G(\nu^*) \leq \phi(G(\zeta), \lambda^d) \cdot \|f\| G(\psi). \quad (11)$$

Proof. We use again the Stein's estimation 3, which we rewrite as

$$\lambda^d \cdot |u|_{q(p)} \leq \lambda^{d/p} \cdot |f|_p \leq \lambda^{d/p} \cdot \|f\| G(\psi) \cdot \psi(p), \quad p \in (a, b). \quad (12)$$

We get after dividing both sides of inequality (12) on the $\nu(p)$ and λ^{-d} :

$$\lambda^d \cdot \frac{|u|_q}{\nu(p)} \leq \|f\| G(\psi) \cdot \frac{\lambda^{d/p}}{\zeta(p)}. \quad (13)$$

Tacking supremum of the bide sides of inequality (13) over the variable $p; p \in (a, b)$, and tacking into account the definition of the fundamental function, we conclude:

$$\lambda^d \cdot \|u\| G(\nu(q/(q-1))) \leq \phi(G(\zeta), \lambda^d) \cdot \|f\| G(\psi). \quad (14)$$

The last assertion (14) is equivalent to the proposition of theorem 2. \square

3. LOW BOUNDS.

In this section we built some examples in order to illustrate the exactness of upper estimations. It is sufficient to consider only the one - dimensional case: $d = 1$, i.e. $x, y \in R^1$.

We choose here the phase function Φ_1 such that

$$\Phi_1 = \Phi_1^{(0)}(x, y) = xy, \quad (x, y) \in [-1, 1]$$

and

$$\Phi_2 = \Phi_2^{(0)}(x, y) = 1, \quad (x, y) \in [-1, 1].$$

Let us denote for the quoted values $p, q(p), f \in L_p, f \neq 0$

$$W(\lambda, f, p) = \frac{|T_\lambda f|_q \lambda^{d/q}}{|f|_p}. \quad (15)$$

where as before $p \in (1, 2]$, $q = q(p) = p/(p-1) \in [2, \infty)$.

From the inequality of E.M.Stein (3) follows that

$$\sup_{\lambda \geq 1} \sup_{p \in (1, 2)} \sup_{f \in L_p, f \neq 0} W(\lambda, f, p) \leq A(\Phi_1^0, \Phi_2^0) < \infty. \quad (16).$$

We intend to prove an inverse inequality at the critical points $\lambda \rightarrow \infty$ and $p \rightarrow 2 - 0$.

Theorem 3.

$$\lim_{\lambda \rightarrow \infty} \lim_{p \rightarrow 2-0} \sup_{f \in L_p, f \neq 0} W(\lambda, f, p) \geq A_1(\Phi_1^0, \Phi_2^0) > 0. \quad (17).$$

Proof. Let us consider the function

$$f(y) = f_0(y) = |y|^{-1/2}, \quad |y| \in (0, 1]$$

and $f_0(y) = 0$ when $y = 0$ or $|y| > 1$. We have:

$$|f_0|_p^p = 2 \int_0^1 y^{-p/2} dy = \frac{4}{2-p}, p \in [1, 2),$$

or equally

$$|f_0|_p = [4/(2-p)]^{1/p}. \quad (18).$$

Further, let us investigate the function $u = T_\lambda f$. Auxiliary denotation: $\Lambda = \lambda x$.

$$\begin{aligned} u &= \int_{-1}^1 \exp(i\lambda xy) |y|^{-1/2} dy = 2 \int_0^1 \cos(\Lambda y) y^{-1/2} dy = \\ &= 2\Lambda^{-1/2} \int_0^\Lambda \cos z / \sqrt{z} dz = 2\Lambda^{-1/2} I(\Lambda), \end{aligned}$$

where

$$I(\Lambda) = \int_0^\Lambda z^{-1/2} \cos z dz. \quad (19)$$

It is easy to calculate:

$$I(\Lambda) \asymp \sqrt{\Lambda}, \quad \Lambda \in (0, 1); \quad |I(\Lambda)| \asymp 1, \quad \Lambda \in (0, 1);$$

therefore

$$|u| \asymp 1, \quad \Lambda \in (0, 1); \quad |u| \asymp \Lambda^{-1/2}, \quad \Lambda \geq 1.$$

Further,

$$|u|_q^q \geq C^q \int_{1/\lambda}^\infty (\lambda x)^{-q/2} dx = 2 C^q \lambda^{-1} (q-2)^{-1}. \quad (20)$$

Substituting into the expression for the functional W , we get to the conclusion of theorem 3 after simple computations.

We can generalize the assertion of last assertion on the $G(\psi)$ spaces as follows. Let us denote

$$\psi_0(p) = [4/(2-p)]^{1/p}, \quad p \in (2, \infty), \quad (21)$$

then $f_0(\cdot) \in G(\psi_0)$ and $\|f_0\|_{G(\psi_0)} = 1$.

Theorem 4.

$$\lim_{\lambda \rightarrow \infty} Z(\lambda, \psi_0, f_0) > A_2(\Phi_1^0, \Phi_2^0) > 0. \quad (22).$$

Proof. From theorem 1 follows that $u(\cdot) \in G(\psi_0^{(\lambda)})$ and $\|u\|_{G(\psi_0^{(\lambda)})} = C_2 < \infty$. Let us now estimate the L_r norm of the function u from below.

Since at $x \geq 1/\lambda$

$$u(x) \geq C(\lambda x)^{-1/2},$$

we have for the values $r > 2$:

$$\begin{aligned} |u|_r^r &\geq C \int_1^\infty (\lambda x)^{-r/2} dx = 2C\lambda^{-1}(r-2)^{-1}; \\ |u|_r &\geq C\lambda^{-1/r}(r-2)^{-1/r}. \end{aligned} \quad (23)$$

Choosing the value $r = q(p) = p/(p-1)$, we obtain on the basis of inequality (23):

$$C_1^{-1} \lambda^{1/q} |u|_q / |f|_p \geq \frac{(2-p)^{1/p}}{(q-2)^{1/q}} = (p-1)^{1/p-1},$$

where C_1 does not depend on the p and λ .

As long as

$$\inf_{p \in (1,2)} (p-1)^{1/p-1} = C_2 > 0,$$

we conclude

$$\inf_{\lambda \geq 2} \inf_{p \in (1,2)} \lambda^{1/q} |u|_q / |f|_p \geq C_2(\Phi_1^0, \Phi_2^0, \psi_0) > 0,$$

or equally

$$\inf_{\lambda \geq 2} \frac{\|T_\lambda f_0\| G(\psi_0^{(\lambda)})}{\|f\| G(\psi_0)} \geq C > 0,$$

which is equivalent to the assertion of theorem 4.

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